Noncyclic geometric phase and its non-Abelian generalization

1999 J. Phys. A: Math. Gen. 32 8157

(http://iopscience.iop.org/0305-4470/32/46/312)

J. Phys. A: Math. Gen. **32** (1999) 8157–8171. Printed in the UK PII: S0305-4470(99)03582-9

Noncyclic geometric phase and its non-Abelian generalization

Ali Mostafazadeh

Department of Mathematics, Koç University, Istinye 80860, Istanbul, Turkey

E-mail: `amostafazadeh@ku.edu.tr`

Received 20 April 1999, in final form 18 August 1999

Abstract. We use the theory of dynamical invariants to yield a simple derivation of noncyclic analogues of the Abelian and non-Abelian geometric phases. This derivation relies only on the principle of gauge invariance and elucidates the existing definitions of the Abelian noncyclic geometric phase. We also discuss the adiabatic limit of the noncyclic geometric phase and compute the adiabatic non-Abelian noncyclic geometric phase for a spin-1 magnetic (or electric) quadrupole interacting with a precessing magnetic (electric) field.

1. Introduction

Since the publication of Berry's seminal paper [1] on the adiabatic geometric phase, the concept of geometric phase has been generalized in a number of ways. Following the work of Aharonov and Anandan [2] on nonadiabatic geometric phase, Samuel and Bhandari [3] showed that one could indeed define an analogue of the Abelian geometric phase for a quantum state that does not undergo a cyclic evolution. Zak [4], Aitchison and Wanelik [5], Mukunda and Simon [6], Pati [7, 8] and de Polavieja and Sjöqvist [9] have ellaborated on the theoretical aspects of noncyclic geometric phases, and Wu and Li [10], Weinfurter and Badurek [11], Christian and Shimony [12], Wagh and Rakhecha [13] and Wagh *et al* [14] have explored its experimental consequences. In all these investigations the authors consider Abelian noncyclic geometric phases. The main purpose of this paper is to offer an alternative approach to noncyclic geometric phases which clarifies the existing results on the Abelian noncyclic geometric phase and allows for its non-Abelian generalization.

The cyclic geometric phase can be conveniently discussed within the framework of the theory of dynamical invariants of Lewis and Riesenfeld [15]. The application of dynamical invariants in the study of the cyclic geometric phases has been considered by Morales [16] and Monteoliva *et al* [17] for the Abelian case and by the present author [18] for the non-Abelian case.

In this paper, we shall first present a brief review of the necessary results from the theory of dynamical invariants and comment on their relevance to the cyclic geometric phases in section 2. In section 3, we outline an alternative approach to the nonadiabatic cyclic geometric phase which is essentially the same as Berry's approach to the adiabatic geometric phase [1]. This generalizes the analysis of [19] in which such an approach is developed for the description of the nonadiabatic cyclic geometric phase of a magnetic dipole interacting with a precessing magnetic field. In section 4, we derive an expression for the evolution operator and discuss its gauge invariance. In section 5, we give our definition of the noncyclic geometric phase and explore its relationship to the cyclic geometric phase [18, 20]. In section 6, we restrict

0305-4470/99/468157+15$30.00

ourselves to the Abelian case and compare our definition of the noncyclic Abelian geometric phase with the earlier definitions [5]. In section 7, we present a discussion of the adiabatic approximation and show that our analysis reproduces the results of [9] in the adiabatic limit. In section 8, we calculate the adiabatic non-Abelian noncyclic geometric phase for a spin-1 magnetic (or electric) quadrupole interacting with a precessing magnetic (resp. electric) field. Finally, we present our concluding remarks in section 9.

2. Invariant operators and cyclic geometric phase

As shown by Lewis and Riesenfeld [15], one can write the solution of the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H(t)|\psi(t)\rangle \tag{1}$$

for a time-dependent Hamiltonian $H(t)$, as a linear combination of certain eigenvectors of a Hermitian dynamical invariant. The latter is a Hermitian operator $I(t)$ satisfying

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = \mathrm{i}[I(t), H(t)]. \tag{2}$$

We shall assume that both $H(t)$ and $I(t)$ have discrete spectra.

Now let us label the eigenvalues of $I(t)$ by λ_n and the degree of degeneracy of λ_n by l_n. Furthermore, let $|\lambda_n, a; t\rangle$ be arbitrary orthonormal eigenvectors of $I(t)$ satisfying

$$I(t)|\lambda_n, a; t\rangle = \lambda_n|\lambda_n, a; t\rangle \tag{3}$$

$$\langle\lambda_m, b; t|\lambda_n, a; t\rangle = \delta_{mn}\delta_{ba} \tag{4}$$

$$\sum_n \sum_{a=1}^{l_n} |\lambda_n, a; t\rangle\langle\lambda_n, a; t| = 1 \tag{5}$$

where a and b are degeneracy labels taking their values in $\{1, 2, \ldots, l_n\}$.

Clearly, unlike the eigenvalues λ_n and the corresponding degeneracy subspaces $\mathcal{H}_{\lambda_n}(t)$, the eigenvectors $|\lambda_n, a; t\rangle$ are not uniquely determined by the eigenvalue equation (3). They are only determined up to unitary transformations of the degeneracy subspaces $\mathcal{H}_{\lambda_n}(t)$,

$$|\lambda_n, a; t\rangle \to |\lambda_n, a; t\rangle' = \sum_{b=1}^{l_n} |\lambda_n, b; t\rangle u_{ba}(t) \tag{6}$$

where $u_{ab}(t)$ are the entries of an arbitrary unitary $l_n \times l_n$ matrix $u(t)$.

The main result of Lewis and Riesenfeld [15] is that one can choose a particular set of eigenvectors $|\lambda_n, a; t\rangle'$ that are solutions of the Schrödinger equation (1). These eigenvectors are given by [18]:

$$|\lambda_n, a; t\rangle' := \sum_{b=1}^{l_n} |\lambda_n, a; t\rangle u^n_{ab}(t) \tag{7}$$

where

$$u^n(t) := \mathcal{T}\mathrm{e}^{-\mathrm{i}\int_0^t \Delta^n(t')\,\mathrm{d}t'} u^n(0) = \mathcal{T}\mathrm{e}^{-\mathrm{i}\int_0^t [\mathcal{E}^n(t') - \mathcal{A}^n(t')]\,\mathrm{d}t'} u^n(0). \tag{8}$$

$\mathcal{T}$ stands for the time-ordering operator and $\Delta^n(t)$, $\mathcal{E}^n(t)$ and $\mathcal{A}^n(t)$ are Hermitian $l_n \times l_n$ matrices with entries

$$\Delta^n_{ab}(t) := \mathcal{E}^n_{ab}(t) - \mathcal{A}^n_{ab}(t) \tag{9}$$

$$\mathcal{E}^n_{ab}(t) := \langle\lambda_n, a; t|H(t)|\lambda_n, b; t\rangle \tag{10}$$

$$\mathcal{A}^n_{ab}(t) := \mathrm{i}\left\langle\lambda_n, a; t\left|\frac{\mathrm{d}}{\mathrm{d}t}\right|\lambda_n, b; t\right\rangle \tag{11}$$

respectively. In other words, $u^n(t)$ is a solution of the matrix Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}u^n(t) = \Delta^n(t)u^n(t). \tag{12}$$

If the matrices $\mathcal{E}^n(t)$ and $\mathcal{A}^n(t)$ commute, then we can write

$$u^n(t) = \mathcal{T}\mathrm{e}^{-\mathrm{i}\int_0^t \mathcal{E}^n(t')\,\mathrm{d}t'}\mathcal{T}\mathrm{e}^{-\mathrm{i}\int_0^t \mathcal{A}^n(t')\,\mathrm{d}t'}u^n(0). \tag{13}$$

For the case where the invariant $I(t)$ is periodic [17, 18], i.e. $I(T) = I(0)$ for some T, $|\lambda_n, a; T\rangle = |\lambda_n, a; 0\rangle$ and $|\lambda_n, a; t\rangle'$ undergoes a cyclic evolution. The corresponding non-Abelian nonadiabatic cyclic geometric phase [20] is given by $\Gamma^n(T)$ where $\Gamma^n(t)$ is defined to be the unique solution of the matrix Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Gamma^n(t) = -\mathcal{A}^n(t)\Gamma^n(t) \qquad \Gamma^n(0) = 1. \tag{14}$$

Alternatively,

$$\Gamma^n(t) := \mathcal{T}\mathrm{e}^{\mathrm{i}\int_0^t \mathcal{A}^n(t')\,\mathrm{d}t'}. \tag{15}$$

If the eigenvalue λ_n is nondegenerate, then $l_n = 1$ and we have

$$u^n(t) = \mathrm{e}^{\mathrm{i}\delta_n(t)}\Gamma^n(t) \tag{16}$$

where

$$\delta_n(t) := -\int_0^t \mathcal{E}^n(t')\,\mathrm{d}t' = -\int_0^t \langle\lambda_n; t'|H(t')|\lambda_n; t'\rangle\,\mathrm{d}t' \tag{17}$$

$$\Gamma^n(t) = \mathrm{e}^{\mathrm{i}\gamma_n(t)} \qquad \text{and} \qquad \gamma_n(t) := \int_0^t \mathcal{A}^n(t')\,\mathrm{d}t' = \int_0^t i\left\langle\lambda_n; t'\left|\frac{\mathrm{d}}{\mathrm{d}t'}\right|\lambda_n; t'\right\rangle\mathrm{d}t'. \tag{18}$$

In this case $\Gamma^n(T)$ is the Abelian nonadiabatic cyclic geometric phase [2].

3. An alternative approach to nonadiabatic cyclic geometric phase

In order to make the geometric character of the cyclic geometric phase more transparent, we shall express the invariant $I(t)$ as a linear combination of a set of linearly independent constant Hermitian operators X_i,

$$I(t) = \sum_{i=1}^{N}\theta^i(t)X_i. \tag{19}$$

Here N is a fixed non-negative integer, the coefficients θ^i are real-valued functions, and X_i are generators of the group $U(\mathcal{H})$ of unitary transformations of the Hilbert space $\mathcal{H}$. If the system has a finite-dimensional dynamical group G, then X_i are the representations of the generators of G. In this case N is just the dimension of G. However, if the Hilbert space is infinite-dimensional, then in principle one will need an infinite number N of generators X_i of $U(\mathcal{H})$ to satisfy (19)†, and one must find a way to make the right-hand side of (19) well defined. We shall not be concerned with the subtleties of the infinite-dimensional unitary group [21], and assume that N is finite.

Now, since the time dependence of $I(t)$ is governed by those of the parameters θ^i, we can consider the parameter-dependent operator $I[\theta]$ with eigenvalues‡ λ_n and eigenvectors $|\lambda_n, a; \theta\rangle$, and write

$$I(t) = I[\theta(t)] \qquad \text{and} \qquad |\lambda_n, a; t\rangle = |\lambda_n, a; \theta(t)\rangle. \tag{20}$$

† Note that for each value of t there is a finite number N of generators X_i for which (19) holds. However, for an infinite-dimensional Hilbert space, as t changes N might not have an upper bound. Therefore, in order to satisfy (19) one would, in general, need to include an infinite number of generators of the group $U(\mathcal{H})$.

‡ Note that the eigenvalues of an invariant operator are constant [15, 18].

Here θ stands for $(\theta^1, \theta^2, \ldots, \theta^N)$. The parameters θ^i may be viewed as local coordinates of a parameter space $\mathcal{M}$. As time progresses they trace a curve $\mathcal{C}$ in $\mathcal{M}$.

If the system possesses a dynamical group G, then we can introduce the parameter-dependent Hamiltonian

$$H[R] = \sum_{i=1}^{N} R^i X_i \tag{21}$$

and identify $H(t)$ with $H[R(t)]$ [1]. This means that the parameter space $\mathcal{M}$ of the invariant (19) is the same as the parameter space of the Hamiltonian (21). In this case, we can use the results of [22] to identify $\mathcal{M}$ with a submanifold of the flag manifold G/T where T is a maximal torus of G.

Now suppose that the curve $\mathcal{C}$ traced by the parameters $\theta(t)$ lies in a local coordinate patch of the parameter space $\mathcal{M}$. In this case, we can introduce the nonadiabatic analogue of the non-Abelian Berry connection one-form [19, 23],

$$A^n[\theta] = \sum_{i=1}^{N} A_i^n[\theta]\, \mathrm{d}\theta^i. \tag{22}$$

The matrix elements of $A^n[\theta]$ and its components $A_i^n[\theta]$ are given by

$$A_{ab}^n[\theta] := \mathrm{i}\langle \lambda_n, a; \theta | d | \lambda_n, b; \theta\rangle \tag{23}$$

$$(A_i^n[\theta])_{ab} := \mathrm{i}\left\langle \lambda_n, a; \theta \left| \frac{\partial}{\partial\theta^i} \right| \lambda_n, b; \theta \right\rangle \tag{24}$$

respectively. In equation (23) d stands for the exterior derivative with respect to θ^i. In view of equations (11), (20), (22), (23), and (15), we have

$$\mathcal{A}^n(t)\, \mathrm{d}t = \sum_{i=1}^{N} A_i^n[\theta(t)] \frac{\mathrm{d}\theta^i(t)}{\mathrm{d}t}\, \mathrm{d}t = A^n[\theta(t)] \tag{25}$$

$$\Gamma^n(t) = \mathcal{P}\mathrm{e}^{\mathrm{i}\int_{\theta(0)}^{\theta(t)} A^n[\theta]} \tag{26}$$

where $\mathcal{P}$ stands for the path-ordering operator. In particular, for a periodic invariant, the curve $\mathcal{C}$ traced by $\theta(t)$ is closed, and the non-Abelian cyclic geometric phase takes the form

$$\Gamma^n(T) = \mathcal{P}\mathrm{e}^{\mathrm{i}\oint_{\mathcal{C}} A^n[\theta]}. \tag{27}$$

As pointed out in [18], this expression agrees with Anandan's definition of non-Abelian cyclic geometric phase [20].

If $\mathcal{C}$ does not lie in a single coordinate patch of $\mathcal{M}$, one must evaluate the path-ordered integrals in (26) and (27) along the segments of $\mathcal{C}$ belonging to different coordinate patches and multiply the resulting unitary matrices in the order in which the curve $\mathcal{C}$ is traversed in time.

4. Evolution operator and its gauge invariance

In general, we can write any solution of the Schrödinger equation (1) as a linear combination of $|\lambda_n, b; t\rangle'$, i.e.,

$$|\psi(t)\rangle = \sum_n \sum_{a=1}^{l_n} \tilde{C}_a^n |\lambda_n, a; t\rangle' = \sum_n \sum_{a,b=1}^{l_n} C_a^n u_{ba}^n(t) |\lambda_n, b; t\rangle \tag{28}$$

where

$$\tilde{C}_a^n := \sum_{b=1}^{l_n} u_{ba}^{n\dagger}(0)\langle\lambda_n, b; t|\psi(0)\rangle \qquad \text{and} \qquad C_a^n := \langle\lambda_n, a; t|\psi(0)\rangle. \quad (29)$$

In view of equation (28), the evolution operator is given by

$$U(t) = \sum_n \sum_{a,b=1}^{l_n} u_{ab}^n(t)|\lambda_n, a; t\rangle\langle\lambda_n, b; 0|. \quad (30)$$

Now let us recall that the eigenvectors $|\lambda_n, a; \theta\rangle$ are not uniquely determined by the eigenvalue equation

$$I[\theta]|\lambda_n, a; \theta\rangle = \lambda_n|\lambda_n, a; \theta\rangle. \quad (31)$$

They are subject to arbitrary gauge transformations

$$|\lambda_n, a; \theta\rangle \to |\lambda_n, a; \theta\rangle\tilde{} := \sum_{b=1}^{l_n} |\lambda_n, b; t\rangle v_{ba}^n[\theta] \quad (32)$$

where $v_{ab}^n[\theta]$ are entries of an $l_n \times l_n$ unitary matrix $v^n[\theta]$. In fact, if we denote the local coordinate patch corresponding to the coordinates θ^i by $\mathcal{O}$, v^n may be viewed as a smooth function mapping $\mathcal{O}$ to the unitary group $U(l_n)$.

Using equation (23), we can easily derive the gauge transformation law for $A^n[\theta]$, namely

$$A^n[\theta] \to \tilde{A}^n[\theta] = v^n[\theta]^\dagger A^n[\theta]v^n[\theta] + \mathrm{i}v^n[\theta]^\dagger\, \mathrm{d}v^n[\theta]. \quad (33)$$

Furthermore, in view of equations (25), (14), and (33), we have the following transformation rules for $\Gamma^n(t)$ and $u^n(t)$:

$$\Gamma^n(t) \to \tilde{\Gamma}^n(t) = v^n[\theta(t)]^\dagger \Gamma^n(t)v^n[\theta(0)] \quad (34)$$

$$u^n(t) \to \tilde{u}^n(t) = v^n[\theta(t)]^\dagger u^n(t)v^n[\theta(0)]. \quad (35)$$

A simple consequence of equations (32) and (35) is that the evolution operator (30) is invariant under the gauge transformations.

For a periodic invariant $I(t)$, with $\theta(T) = \theta(0)$, $\tilde{\Gamma}^n(T)$ is related to $\Gamma^n(T)$ by a similarity transformation

$$\tilde{\Gamma}^n(T) = v^n[\theta(0)]^\dagger \Gamma^n(T)v^n[\theta(0)]. \quad (36)$$

In other words, under a gauge transformation (32) $\Gamma^n(T)$ transforms covariantly. Therefore, its eigenvalues and, in particular, its trace are gauge invariant. These are essentially the physically observable quantities associated with the non-Abelian cyclic geometric phase.

5. Noncyclic geometric phase

The main reason for Berry's consideration of a periodic Hamiltonian [1] and Aharonov and Anandan's consideration of cyclic evolutions [2] is the fact that for a cyclic state with period T—in our approach a T-periodic dynamical invariant—the unitary matrix $\Gamma^n(T)$ transforms convariantly under a gauge transformation (32). This property of $\Gamma^n(T)$ guarantees that its eigenvalues and its trace are gauge invariant. This in turn raises the issue of exploring their physical consequences. There is, however, another way of constructing gauge-invariant quantities using $\Gamma^n(t)$.

In view of equations (34) and (35), $\Gamma^n(t)$ and $u^n(t)$ have the same gauge transformation properties. This means that if we replace $u^n(t)$ in the expression (30) for the evolution operator by $\Gamma^n(t)$, then we shall still obtain a gauge-invariant operator, namely

$$V(t) := \sum_n \sum_{a,b=1}^{l_n} \Gamma^n_{ab}(t)|\lambda_n, a; \theta(t)\rangle\langle\lambda_n, b; \theta(0)|. \tag{37}$$

In fact, because the gauge transformations (32) are the unitary transformations of the degeneracy subspaces $\mathcal{H}_{\lambda_n}[\theta(t)]$, the restriction (or projection) of $V(t)$ onto the degeneracy subspaces, i.e.,

$$V^n(t) := \sum_{a,b=1}^{l_n} \Gamma^n_{ab}(t)|\lambda_n, a; \theta(t)\rangle\langle\lambda_n, b; \theta(0)| \tag{38}$$

will also be gauge invariant. By construction, the operators $V^n(t)$ are uniquely determined by the curve $\mathcal{C}$ and its end points. In particular, they are independent of the duration of the evolution. Therefore, they are also geometric quantities.

Next let us note that the gauge invariance and geometric character of $V^n(t)$ will not be affected, if we exchange the positions of $|\lambda_n, a; \theta(t)\rangle$ and $\langle\lambda_n, b; \theta(0)|$ in equation (38). In this way we obtain a set of gauge-invariant scalars:

$$\Pi^n(t) := \sum_{a,b=1}^{l_n} \Gamma^n_{ab}(t)\langle\lambda_n, b; \theta(0)|\lambda_n, a; \theta(t)\rangle = \sum_{a,b=1}^{l_n} w^n_{ba}(t)\Gamma^n_{ab}(t) = \mathrm{trace}[w^n(t)\Gamma^n(t)] \tag{39}$$

where we have introduced the $l_n \times l_n$ matrices $w^n(t)$ with entries

$$w^n_{ba}(t) := \langle\lambda_n, b; \theta(0)|\lambda_n, a; \theta(t)\rangle. \tag{40}$$

By definition $w^n(t)$ only depend on the end points of the curve $\mathcal{C}$. $\Gamma^n(t)$ are also uniquely determined by $\mathcal{C}$. Therefore, as expected, $\Pi^n(t)$ are geometric quantities.

Now let us consider a T-periodic invariant (a cyclic evolution), for which $|\lambda_n, a; \theta(T)\rangle = |\lambda_n, a; \theta(0)\rangle$. In this case, $w^n(T) = w^n(0)$ is just the identity matrix, and equation (39) reduces to

$$\Pi^n(T) = \mathrm{trace}[\Gamma^n(T)]. \tag{41}$$

Therefore, for a cyclic evolution $\Pi^n(T)$ yields the trace of the non-Abelian cyclic geometric phase [23, 24].

On the other hand, since $\Pi^n(t)$ is gauge invariant, we can compute it in a basis $\{|\lambda_n, a; \theta(t)\rangle_\star\}$ in which $\Gamma^n(t)$ is diagonal. If we denote the eigenvalues of $\Gamma^n(t)$ by $\mathrm{e}^{\mathrm{i}\gamma^a_n(t)}$, then we have

$$\Pi^n(t) = \sum_{a=1}^{l_n} \mathrm{e}^{\mathrm{i}\gamma^a_n(t)}\, {}_\star\langle\lambda_n, a; \theta(t)|\lambda_n, a; \theta(0)\rangle_\star. \tag{42}$$

As seen from equation (42), $\Pi^n(t)$ is, in general, a nonunimodular complex number.

In view of the above analysis, the matrix $\check{\Gamma}^n(\mathcal{C})$ defined by

$$\check{\Gamma}^n(\mathcal{C}) = \check{\Gamma}^n(t) := w^n(t)\Gamma^n(t) \tag{43}$$

is a gauge-covariant geometric quantity. We shall therefore identify it as the *non-Abelian noncyclic geometric phase factor*. Clearly, the eigenvalues of $\check{\Gamma}^n(\mathcal{C})$ and its trace namely $\Pi^n(t)$ are gauge-invariant quantities which can, in principle, be observed experimentally. The non-Abelian noncyclic geometric phase factor $\check{\Gamma}^n(\mathcal{C})$ is therefore as physically significant as its cyclic counterpart (27).

As mentioned above, for a cyclic evolution where the invariant $I(t)$ is T-periodic, $|\lambda_n, a; \theta(T)\rangle = |\lambda_n, a; \theta(0)\rangle$ and $w^n(T)$ is the identity operator. In this case, $\check{\Gamma}^n(T)$ is identical with the cyclic geometric phase factor $\Gamma^n(T)$ given by equation (27).

6. Abelian noncyclic geometric phase

For a nondegenerate eigenvalue λ_n of the invariant $I(t)$, we have

$$\check{\Gamma}^n(t) = w^n(t)\Gamma^n(t) = \Pi^n(t) = \langle\lambda_n; \theta(0)|\lambda_n; \theta(t)\rangle \mathrm{e}^{\mathrm{i}\gamma_n(t)} \tag{44}$$

where $\gamma_n(t)$ is the phase angle given by (18). In particular,

$$|\check{\Gamma}^n(t)| = |w^n(t)| = |\langle\lambda_n; \theta(0)|\lambda_n; \theta(t)\rangle| \tag{45}$$

depends only on the end points of the curve $\mathcal{C}$. If $|\check{\Gamma}^n(t)| \neq 0$, then we can consider the phase of $\check{\Gamma}^n(t)$ which is given by

$$\mathrm{e}^{\mathrm{i}\check{\gamma}_n(t)} := \frac{\check{\Gamma}^n(t)}{|\check{\Gamma}^n(t)|} = \mathrm{e}^{\mathrm{i}[\eta_n(t)+\gamma_n(t)]} \tag{46}$$

where

$$\mathrm{e}^{\mathrm{i}\eta_n(t)} := \frac{w^n(t)}{|w^n(t)|} = \frac{\langle\lambda_n; \theta(0)|\lambda_n; \theta(t)\rangle}{|\langle\lambda_n; \theta(0)|\lambda_n; \theta(t)\rangle|} = \left[\frac{\langle\lambda_n; \theta(0)|\lambda_n; \theta(t)\rangle}{\langle\lambda_n; \theta(t)|\lambda_n; \theta(0)\rangle}\right]^{1/2}. \tag{47}$$

The phase angle $\check{\gamma}_n(t)$ is a *real noncyclic geometric phase angle*. It consists of two pieces: $\gamma_n(t)$ that depends on the curve $\mathcal{C}$, and $\eta_n(t)$ that depends on the end points of $\mathcal{C}$.

The phase factor (46) coincides with the 'real noncyclic geometric phase' introduced by Aitchison and Wanelik [5]. As discussed by Aitchison and Wanelik it is equivalent to the noncyclic geometric phase of Samuel and Bhandari [3] and Mukunda and Simon [6].

7. Adiabatic approximation and the noncyclic geometric phase in the adiabatic limit

Let $H[R]$ be a parameter-dependent Hamiltonian with a discrete spectrum. We shall denote the eigenvalues of $H[R]$ by $E_n[R]$, their degree of degeneracy by $\mathcal{N}$, and the corresponding degeneracy subspaces by $\mathcal{H}_n[R]$. Let $|n, a; R\rangle$ form a complete set of orthonormal eigenvectors of $H[R]$. They satisfy,

$$H[R]|n, a; R\rangle = E_n[R]|n, a; R\rangle \tag{48}$$

where a is a degeneracy label taking its value in $\{1, 2, \ldots, \mathcal{N}\}$.

Now consider the time-dependent Hamiltonian $H(t) := H[R(t)]$, where the parameters $R(t) = (R^1(t), R^2(t), \ldots, R^d(t))$ trace a curve C in the parameter space M of the Hamiltonian. We shall denote the duration of the evolution of the system by τ and suppose that during the evolution no level crossings occur. Furthermore, let $I(t)$ be a dynamical invariant satisfying (2), and suppose that it depends on a set of parameters $\theta = (\theta^1, \theta^2, \ldots, \theta^N)$, i.e., $I(t) = I[\theta(t)]$ where $\theta(t)$ traces a curve $\mathcal{C}$ in the parameter space $\mathcal{M}$ of the invariant. Since $I(t)$ yields the solution of the Schrödinger equation, the dynamics of the system can be encoded in the definition of a function

$$F : M \to \mathcal{M} \qquad \text{defined by} \quad F(R) := \theta. \tag{49}$$

In particular, F maps the curve C onto the curve $\mathcal{C}$, and $I(t) = I[\theta(t)] = I[F(R(t))]$. Note, however, that there does not exist a universal function F describing all possible dynamical processes. In other words, the definition of F also depends on the choice of the Hamiltonian or alternatively the curve C. In this sense, it is more appropriate to define the function

$$\mathcal{F} : \mathcal{P}_M \to \mathcal{P}_{\mathcal{M}} \qquad \text{by} \quad \mathcal{F}(C) = \mathcal{C} \tag{50}$$

where $\mathcal{P}_M$ and $\mathcal{P}_{\mathcal{M}}$ are the space of paths in M and $\mathcal{M}$ respectively. The latter are infinite-dimensional spaces. Therefore, it is more convenient to use the function F with the provision of its nonuniversal character.

Next we define a normalized time variable by $s := t/\tau$, and assume that for sufficiently large values of τ we can expand $I(t)$ and $H(t)$ in powers of τ^{-1}, i.e.,

$$I(t) = I(\tau s) = I_0(s) + \sum_{\ell=1}^{\infty} \tau^{-\ell} I_\ell(s) \qquad \text{and} \qquad H(t) = H(\tau s) = H_0(s) + \sum_{\ell=1}^{\infty} \tau^{-\ell} H_\ell(s) \tag{51}$$

where I_ℓ and H_ℓ are Hermitian operators and $I_0 \neq 0 \neq H_0$. If we substitute $t = \tau s$ and (51) in (2) and take the limit $\tau \to \infty$, we find $[I_0(s), H_0(s)] = 0$. This means that in this limit, where $I(t) \to I_0(s)$ and $H(t) \to H_0(s)$, the eigenvectors of $I(t)$ and $H(t)$ coincide. Using the notation of the preceding sections, we have

$$\lim_{\tau\to\infty} |\lambda_n, a; t\rangle = |n, a; t\rangle. \tag{52}$$

Because $|\lambda_n, a; t\rangle = |\lambda_n, a; \theta(t)\rangle$, $|n, a; t\rangle = |n, a; R(t)\rangle$, and (52) is independent of the form of the curve C,

$$\lim_{\tau\to\infty} |\lambda_n, a; \theta\rangle|_{\mathcal{C}} = |n, a; R\rangle|_C. \tag{53}$$

In particular, this implies that for sufficiently large values of τ, we can choose an invariant whose parameter space is the same as that of the Hamiltonian, $\mathcal{M} = M$. Therefore, C and $\mathcal{C}$ belong to the same parameter space M. In this case, we can also express (52) and (53) by

$$\lim_{\tau\to\infty} F = \mathrm{id}_M \tag{54}$$

where id_M is the identity function on M.

Now let us use equations (30) and (52) to compute the evolution operator in the limit $\tau \to \infty$. This leads to

$$\lim_{\tau\to\infty} U(t) = U^{(0)}(t) \tag{55}$$

where

$$U^{(0)}(t) := \sum_n \sum_{a,b=1}^{\mathcal{N}} u^n_{\circ ab}(t) |n, a; t\rangle\langle n, b; 0| \tag{56}$$

$$u^n_\circ(t) = \mathrm{e}^{\mathrm{i}\delta_{\circ n}(t)} \Gamma^n_\circ(t) \tag{57}$$

$$\delta_{\circ n}(t) := -\int_0^t E_n(t')\,\mathrm{d}t' \tag{58}$$

$$\Gamma^n_\circ(t) := \mathcal{P}\mathrm{e}^{\mathrm{i}\int_{R(0)}^{R(t)} A^n_\circ[R]} \tag{59}$$

$$(A^n_\circ[R])_{ab} := \mathrm{i}\langle n, a; R|d|n, b; R\rangle. \tag{60}$$

Here the subscript $\circ$ is inserted to mean that the corresponding quantities are obtained in the adiabatic limit ($\tau \to \infty$) from the ones without a subscript $\circ$. The matrix of one-forms $A^n_\circ[R]$ is the non-Abelian Berry connection one-form [23].

In practice, τ is a finite number and the limit $\tau \to \infty$ is interpreted by the condition that τ must be much larger than the time (inverse of energy) scale of the quantum system.

If this happens to be the case the above results may be used. It is not difficult to check that an operator $I(t)$ with the same eigenvectors as the Hamiltonian satisfies equation (2) only if the eigenvectors of the Hamiltonian are constant. This is often not the case. Therefore, in an eigenbasis $\{|n, a; t\rangle\}$ of the Hamiltonian, the invariant $I(t)$ is not diagonal. However, if the above adiabaticity condition is fulfilled, i.e., τ is much larger than the timescale of the problem, then the off-diagonal matrix elements of $I(t)$ are much smaller than its diagonal matrix elements. The approximation scheme in which one neglects the off-diagonal matrix elements of $I(t)$ is called the *adiabatic approximation*, [25–27]. In this approximation, we have

$$|\lambda_n, a; t\rangle \approx |n, a; t\rangle \qquad U(t) \approx U^{(0)}(t) \qquad \text{and} \qquad F \approx \mathrm{id}_M. \tag{61}$$

This is a valid approximation scheme if and only if

$$\mathcal{A}_{ba}^{mn}(t) := \mathrm{i}\left\langle m, b; R \left| \frac{\mathrm{d}}{\mathrm{d}t} \right| n, a; R \right\rangle = \frac{\mathrm{i}\langle m, b; R|\frac{\mathrm{d}H(t)}{\mathrm{d}t}|n, a; R\rangle}{E_n(t) - E_m(t)} \approx 0 \qquad \text{for} \quad m \neq n. \tag{62}$$

Here m and n are arbitrary labels (satsifying $m \neq n$) and a and b are arbitrary degeneracy labels associated with the eigenvalues $E_n(t)$ and $E_m(t)$, respectively. The second equation in (62) is obtained by differentiating both sides of equation (48) with respect to time and taking the inner product of both sides of the resulting equation with $|m, b; t\rangle$. The meaning of '≈ 0' in (62) is that the left-hand side of (62) which has the dimension of frequency must be much smaller than the frequency (energy) scale of the system.

In view of (61), we have the following expression for the adiabatic non-Abelian noncyclic geometric phase (43):

$$\check{\Gamma}_\circ^n(t) = w_\circ^n(t)\Gamma_\circ^n(t) \tag{63}$$

where $w_\circ^n(t)$ is an $\mathcal{N} \times \mathcal{N}$ matrix with entries

$$w_{\circ ab}^n(t) := \langle n, a; R(0)|n, b; R(t)\rangle. \tag{64}$$

If $E_n[R]$ is nondegenerate, $\mathcal{N} = 1$ and

$$\check{\Gamma}_\circ^n(t) = w_\circ^n(t)\mathrm{e}^{\mathrm{i}\gamma_{\circ n}(t)} \qquad \text{where} \quad \gamma_{\circ n}(t) := \int_{R(0)}^{R(t)} A_\circ^n[R] = \int_{R(0)}^{R(t)} \mathrm{i}\langle n; R|d|n; R\rangle. \tag{65}$$

The phase of $\check{\Gamma}_\circ^n(t)$, namely

$$\mathrm{e}^{\mathrm{i}\check{\gamma}_{\circ n}(t)} = \left[\frac{\langle n; R(0)|n; R(t)\rangle}{\langle n; R(t)|n; R(0)\rangle}\right]^{1/2} \mathrm{e}^{\mathrm{i}\gamma_{\circ n}(t)} \tag{66}$$

is precisely the Abelian adiabatic noncyclic geometric phase studied by de Polavieja and Sjöqvist [9].

8. Application: spin-1 quadrupole in a precessing magnetic field

The simplest possible quantum system that allows for the occurrence of a non-Abelian geometric phase is a system with a three-dimensional Hilbert space and a dynamical invariant $I(t)$ which has a nondegenerate and a degenerate eigenvalue [18]. If the system undergoes an adiabatic evolution, then the role of the invariant is essentially played by the Hamiltonian. In particular, the Hamiltonian must have a nondegenerate and a degenerate eigenvalue. The moduli space of all such Hamiltonians (for a three-dimensional Hilbert space) has the manifold structure of the projective space $\mathbb{C}P^2$, [23]. A thorough treatment of the problem of the adiabatic non-Abelian cyclic geometric phase for such a system is presented in [28]. In this section, we shall use the results of [28] to investigate the adiabatic non-Abelian noncyclic geometric phase $\check{\Gamma}^n_\circ(t)$ for a spin-1 quadrupole interacting with a precessing magnetic or electric field. The problem of the non-Abelian geometric phase for a spin-$\frac{3}{2}$ quadrupole has been considered by Zee [24], Mead [29] and Avron *et al* [30, 31]. For a bosonic system such as the spin-1 quadrupole considered here, one can show that Berry's connection one-form is a pure gauge and Berry's cyclic geometric phase angle vanishes [31]. This result does not, however, generalize to the non-Abelian geometric phase [28].

Consider the quadrupole Stark Hamiltonian $H = \lambda(J\cdot R)^2$, where λ is a real coupling constant, $J = (J_1, J_2, J_3)$ is the angular momentum operator, and $R = (R^1, R^2, R^3)$ is a 3-vector representing the magnetic (or the electric) field. For a spin-1 particle, this Hamiltonian has the form

$$H[R] = \frac{\lambda\rho}{2}\begin{pmatrix} 1+2\zeta^2 & \sqrt{2}\zeta\,\mathrm{e}^{-\mathrm{i}\varphi} & \mathrm{e}^{-2\mathrm{i}\varphi} \\ \sqrt{2}\zeta\,\mathrm{e}^{\mathrm{i}\varphi} & 2 & -\sqrt{2}\zeta\,\mathrm{e}^{-\mathrm{i}\varphi} \\ \mathrm{e}^{-2\mathrm{i}\varphi} & -\sqrt{2}\zeta\,\mathrm{e}^{\mathrm{i}\varphi} & 1+2\zeta^2 \end{pmatrix} \tag{67}$$

where (ρ, φ, z) are the cyclindrical coordinates in the R-space, i.e.,

$$\rho := \sqrt{(R^1)^2+(R^2)^2} \qquad \varphi := \tan^{-1}(R^2/R^1) \qquad z := R^3$$

$\zeta := z/\rho$, and we have used the standard spin-1 representations of J_i.

In view of the general results of [28], the eigenvalues of the Hamiltonian (67) are given by

$$E_1[R] = \lambda\rho^2\zeta^2 \qquad \text{and} \qquad E_2[R] = \lambda\rho^2(1+\zeta^2) \tag{68}$$

where $R = (\rho, \varphi, \zeta)$. As seen from (68), if $\rho \neq 0$, then the Hamiltonian has two distinct eigenvalues. In this case, $E_1[R]$ is nondegenerate and $E_2[R]$ is doubly degenerate. A set of orthonormal eigenvectors of $H[R]$ is given by [28]

$$|1; R\rangle := N_1^{-1}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi} \\ \sqrt{2}\zeta \\ \mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix}$$
$$|2,1; R\rangle := N_2^{-1}\begin{pmatrix} -\sqrt{2}\zeta\,\mathrm{e}^{-\mathrm{i}\varphi} \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |2,2; R\rangle := (N_1N_2)^{-1}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi} \\ \sqrt{2}\zeta \\ (1+2\zeta^2)\mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix} \tag{69}$$

where $N_1 := \sqrt{2(1+\zeta^2)}$ and $N_2 := \sqrt{1+2\zeta^2}$. Note that these formulae are valid for $\rho \neq 0$, i.e., everywhere except the R^3-axis.

Again using the general results of [28] or by direct calculation, we can compute Berry's connection one-forms $A^n_\circ$. Doing the necessary algebra, we find

$$A^1_\circ[R] = \mathrm{d}\varphi \qquad \text{and} \qquad A^2_\circ[R] = \begin{pmatrix} \mu\,\mathrm{d}\varphi & \nu\mathrm{e}^{\mathrm{i}\varphi}\,\mathrm{d}\varphi \\ \nu\mathrm{e}^{-\mathrm{i}\varphi}\,\mathrm{d}\varphi & \sigma\,\mathrm{d}\varphi \end{pmatrix} \tag{70}$$

where

$$\mu := \frac{2\zeta^2}{1+2\zeta^2} = \frac{2c^2}{1+c^2} \tag{71}$$

$$\nu := -\frac{\zeta}{(1+2\zeta^2)\sqrt{1+\zeta^2}} = -\frac{c(1-c^2)}{1+c^2} \tag{72}$$

$$\sigma := -\frac{1+2(1+2\zeta^2)^2}{2(1+2\zeta^2)(1+\zeta^2)} = -\frac{1+2(1+c^2)^2}{2(1+c^2)} \tag{73}$$

and $c := \zeta/(1+\zeta^2) = z/\sqrt{\rho^2+\zeta^2} = R^3/\sqrt{(R^1)^2+(R^2)^2+(R^3)^2}$. Note that in the spherical coordinates (r, θ, φ), we have

$$c = \cos\theta. \tag{74}$$

In view of equations (70), $A^1_\circ[R]$ is a pure gauge. This was to be expected, for the system is bosonic [31]. The connection one-form $A^2_\circ[R]$ is not a pure gauge. In fact, unlike the spin-$\frac{3}{2}$ systems considered in the literature [24] even for the case of a precessing field where θ is kept fixed and φ varies, the adiabatic geometric phase associated with E_2 is non-Abelian.

In the remainder of this section, we shall compute the adiabatic non-Abelian noncyclic geometric phase associated with the degenerate eigenvalue E_2 for a precessing field with

$$\theta = \text{constant} \qquad \varphi = \varphi_0 + \omega t \qquad \text{and} \qquad \varphi_0, \omega = \text{constant}. \tag{75}$$

First, let us consider the matrix $\Gamma^2_\circ$. We can write equation (59) as the matrix Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi}\Gamma^2_\circ(\varphi) = h(\varphi)\Gamma^2_\circ(\varphi) \tag{76}$$

where

$$h(\varphi)\,\mathrm{d}\varphi := -A^2_\circ[R]. \tag{77}$$

Clearly, $h(\varphi)$ belongs to the Lie algebra of the unitary group $U(2)$. In particular, it can be written in the form

$$h(\varphi) = \frac{1}{2}\sum_{\ell=0}^{3} r^\ell \sigma_\ell \tag{78}$$

where σ_0 stands for the unit 2×2 matrix, σ_ℓ with $\ell \in \{1, 2, 3\}$ are Pauli matrices, and

$$r^0 := -(\mu + \sigma) \qquad r^1 := -\nu\cos\varphi$$
$$r^2 := \nu\sin\varphi \qquad \text{and} \qquad r^3 := \sigma - \mu.$$

Substituting these equations in (78), we obtain

$$\begin{aligned} h(\varphi) &= \frac{1}{2}[-(\mu+\sigma)\sigma_0 - \nu(\cos\varphi\sigma_1 - \sin\varphi\,\sigma_2) + (\sigma-\mu)\sigma_3] \\ &= \frac{1}{2}\mathrm{e}^{\mathrm{i}\varphi\sigma_3/2}[-(\mu+\sigma)\sigma_0 - \nu\sigma_1 + (\sigma-\mu)\sigma_3]\mathrm{e}^{-\mathrm{i}\varphi\sigma_3/2} \end{aligned} \tag{79}$$

where we have used the identity

$$\mathrm{e}^{-\mathrm{i}\varphi\sigma_i/2}\sigma_j\mathrm{e}^{\mathrm{i}\varphi\sigma_i/2} = \cos\varphi\sigma_j + \sin\varphi\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k \qquad \text{for} \quad i \neq j. \tag{80}$$

In (80), ϵ_{ijk} stands for the totally antisymmetric Levi-Civita symbol with $\epsilon_{123} = 1$.

In view of equation (79), $h(\varphi)$ is the Hamiltonian of a spin-$\frac{1}{2}$ magnetic dipole in a precessing magnetic field. Therefore, we can perform a unitary transformation of the Hilbert space [33,34] to map it to a constant Hamiltonian. Under a φ-dependent unitary transformation $\mathcal{U}(\varphi)$, $h(\varphi)$ and $\Gamma^2_\circ(\varphi)$ transform according to [34]

$$h(\varphi) \to h'(\varphi) = \mathcal{U}(\varphi)h(\varphi)\mathcal{U}(\varphi)^\dagger - \mathrm{i}\mathcal{U}(\varphi)\frac{\mathrm{d}}{\mathrm{d}\varphi}\mathcal{U}(\varphi)^\dagger \tag{81}$$

$$\Gamma^2_\circ(\varphi) \to \Gamma'^2_\circ(\varphi) = \mathcal{U}(\varphi)\Gamma^2_\circ(\varphi)\mathcal{U}(\varphi_0)^\dagger. \tag{82}$$

Setting $\mathcal{U}(\varphi) = \mathrm{e}^{-\mathrm{i}\varphi\sigma_3/2}$ in (81) and using (79), we find

$$h' = \frac{1}{2}[-(\mu+\sigma)\sigma_0 - \nu\sigma_1 + (1-\mu+\sigma)\sigma_3]. \tag{83}$$

For a precessing field, θ and consequently c, μ, ν, and σ are constant parameters. Therefore, h' is constant, and we have

$$\Gamma'^2_\circ(\varphi) = \mathrm{e}^{-\mathrm{i}h'(\varphi-\varphi_0)}. \tag{84}$$

Substituting this equation in (82), we find

$$\Gamma^2_\circ(\varphi) = \mathrm{e}^{\mathrm{i}\varphi\sigma_3/2}\mathrm{e}^{-\mathrm{i}h'(\varphi-\varphi_0)}\mathrm{e}^{-\mathrm{i}\varphi_0\sigma_3/2}. \tag{85}$$

In view of equations (83) and (85), the matrix elements of $\Gamma^2_\circ(\varphi)$ are given by

$$\Gamma^2_{\circ 11} = \mathrm{e}^{\mathrm{i}(\mu+\sigma+1)(\varphi-\varphi_0)/2}\left\{\cos[(\varphi-\varphi_0)\Delta/2] + \mathrm{i}\left(\frac{\mu-\sigma-1}{\Delta}\right)\sin[(\varphi-\varphi_0)\Delta/2]\right\} \tag{86}$$

$$\Gamma^2_{\circ 12} = \mathrm{i}\left(\frac{\nu}{\Delta}\right)\mathrm{e}^{\mathrm{i}(\mu+\sigma)(\varphi-\varphi_0)/2}\mathrm{e}^{\mathrm{i}(\varphi+\varphi_0)/2}\sin[(\varphi-\varphi_0)\Delta/2] \tag{87}$$

$$\Gamma^2_{\circ 21} = \mathrm{i}\left(\frac{\nu}{\Delta}\right)\mathrm{e}^{\mathrm{i}(\mu+\sigma)(\varphi-\varphi_0)/2}\mathrm{e}^{-\mathrm{i}(\varphi+\varphi_0)/2}\sin[(\varphi-\varphi_0)\Delta/2] \tag{88}$$

$$\Gamma^2_{\circ 22} = \mathrm{e}^{\mathrm{i}(\mu+\sigma-1)(\varphi-\varphi_0)/2}\left\{\cos[(\varphi-\varphi_0)\Delta/2] - \mathrm{i}\left(\frac{\mu-\sigma-1}{\Delta}\right)\sin[(\varphi-\varphi_0)\Delta/2]\right\} \tag{89}$$

where

$$\Delta := \sqrt{(1+\sigma-\mu)^2+\nu^2} = \frac{\sqrt{1+4c^2(4+8c^2+7c^4+c^6)}}{2(1+c^2)}. \tag{90}$$

Next we compute the entries of the matrix $w^2_\circ(t)$ of equation (64). Using equations (69) and doing the necessary algebra, we have

$$w^2_{\circ 11} = 1 - \left(\frac{2c^2}{1+c^2}\right)(1-\mathrm{e}^{-\mathrm{i}(\varphi-\varphi_0)}) = 1-\mu(1-\mathrm{e}^{-\mathrm{i}(\varphi-\varphi_0)}) \tag{91}$$

$$w^2_{\circ 12} = w^2_{\circ 21} = \left[\frac{c(1-c^2)}{1+c^2}\right](1-\mathrm{e}^{-\mathrm{i}(\varphi-\varphi_0)}) = -\nu(1-\mathrm{e}^{-\mathrm{i}(\varphi-\varphi_0)}) \tag{92}$$

$$\begin{aligned} w^2_{\circ 22} &= 1 - \left(\frac{1+c^4}{1+c^2}\right)[1-\cos(\varphi-\varphi_0)] + \mathrm{i}\left(\frac{2c^2}{1+c^2}\right)\sin(\varphi-\varphi_0) \\ &= 1 + \left(\sigma + \frac{3\mu}{4} + \frac{1}{2}\right)[1-\cos(\varphi-\varphi_0)] + \mathrm{i}\mu\sin(\varphi-\varphi_0). \end{aligned} \tag{93}$$

Having obtained $\Gamma^2_\circ$ and $w^2_\circ$, we can use equation (63) to calculate the non-Abelian noncyclic geometric phase $\check{\Gamma}^2_\circ = w^2_\circ\Gamma^2_\circ$. As seen from the above formulae the result will only depend on φ_0, φ, and $c = \cos\theta$. The expression for $\check{\Gamma}^2_\circ$ is rather lengthy. Therefore, we shall instead give its trace $\Pi^2_\circ$ which is of physical importance,

$$\Pi^2_\circ = \mathrm{e}^{\mathrm{i}(\mu+\sigma)(\varphi-\varphi_0)/2}\left(\mathcal{X}\cos\left[\frac{\Delta(\varphi-\varphi_0)}{2}\right] + \mathcal{Y}\sin\left[\frac{\Delta(\varphi-\varphi_0)}{2}\right]\right) \tag{94}$$

where

$$\begin{aligned} \mathcal{X} &:= \tfrac{1}{4}\mathrm{e}^{-\mathrm{i}(\varphi-\varphi_0)/2}[6+7\mu+4\sigma+(2-7\mu-4\sigma)\cos(\varphi-\varphi_0)+4i\sin(\varphi-\varphi_0)] \\ \mathcal{Y} &:= \frac{\mathrm{i}}{8\Delta}\mathrm{e}^{-\mathrm{i}(3\varphi-\varphi_0)/2}(\mathrm{e}^{\mathrm{i}\varphi_0}-\mathrm{e}^{\mathrm{i}\varphi})\{8\nu^2(1+\mathrm{e}^{\mathrm{i}(\varphi+\varphi_0)}) + \mathrm{e}^{\mathrm{i}\varphi_0}[\mu(7\mu-5)-3(2+\mu)\sigma-4\sigma^2-2] \\ &\qquad +\mathrm{e}^{\mathrm{i}\varphi}[\mu(9\mu-19)+(14-13\mu)\sigma+4\sigma^2+10]\}. \end{aligned}$$

One can check that for $\varphi = \varphi_0$, $\Pi^2_\circ = 2$, as expected.

Furthermore, setting $\varphi = \varphi_0 + 2\pi$ in (94), we obtained the trace of the cyclic non-Abelian geometric phase (59) which is given by

$$\Pi^2_\circ|_{\mathrm{cyclic}} = -2\mathrm{e}^{\mathrm{i}\pi(\mu+\sigma)}\cos(\pi\Delta). \tag{95}$$

In this case, we can easily compute the eigenvalues of the non-Abelian cyclic geometric phase. They turn out to be $\mathrm{e}^{\pm\mathrm{i}\pi(\Delta+1)}$.

Now choosing $\varphi_0 = 0$ and $c = \cos\theta = 1/\sqrt{3}$, which are the value used in Tycko's experiment [32], we have $\Delta = \sqrt{889}/24 \approx 1.24$ and

$$\Pi^2_\circ \approx \tfrac{1}{3}\mathrm{e}^{0.10\mathrm{i}\varphi}\{(2 + 4\cos\varphi + 3\mathrm{i}\sin\varphi)\cos(0.62\varphi) \\ +0.81[\cos(\varphi/2) + 2.42\mathrm{i}\sin(\varphi/2)]\sin(\varphi/2)\sin(0.62\varphi)\}.$$

9. Conclusion

In this paper we used the theory of dynamical invariants of Lewis and Riesefeld to develop a general parameter space approach for the nonadiabatic geometric phase. We introduced a set of time-dependent gauge-invariant geometric quantities $\Pi^n(t)$ which we identified with the trace of certain time-dependent gauge-covariant geometric quantities $\check{\Gamma}^n(t)$. For a cyclic evolution with period T, $\check{\Gamma}^n(T)$ yields the non-Abelian cyclic geometric phase. In the Abelian case, $\check{\Gamma}^n(t)/|\check{\Gamma}^n(t)|$ coincides with the Abelian noncyclic geometric phase studied in the literature. Therefore, we identified the $\check{\Gamma}^n(t)$ as a non-Abelian noncyclic geometric phase factor.

We also discussed the adiabatic limit $\check{\Gamma}^n_\circ(t)$ of $\check{\Gamma}^n(t)$. Again we showed that for the Abelian case $\check{\Gamma}^n_\circ(t)/|\check{\Gamma}^n_\circ(t)|$ is the known adiabatic noncyclic geometric phase.

We have finally presented a through analysis of the non-Abelian cyclic and noncyclic geometric phases for a spin-1 quadrupole in a precessing field.

We wish to conclude this paper by pointing out the following observations.

(1) The original definition of the (nonadiabatic) non-Abelian cyclic geometric phase is due to Anandan [20]. As pointed out in [18] Anandan's definition is identical with the one given in terms of the dynamical invariants. More specifically, the basis vectors $|\tilde{\psi}_a(t)\rangle$ used by Anandan [20] to yield the non-Abelian connection one-form are precisely the basis eigenvectors $|\lambda_n, a; \theta(t)\rangle$ of our approach. In fact, our approach may be viewed as a means to identify Anandan's basis vectors $|\tilde{\psi}_a(t)\rangle$ with the eigenvectors of a Hermitian operator, namely a dynamical invariant $I[\theta(t)]$.

(2) In [7, 8], Pati shows that the Abelian noncyclic geometric phase angle of equation (46) may be written in the form

$$\check{\gamma}_n = \int_{\mathcal{C}} \Omega^n \tag{96}$$

where $\Omega^n = \mathrm{i}\langle\chi_n(t)|d|\chi_n(t)\rangle$ and $|\chi_n(t)\rangle$ is a properly scaled state vector. Although Pati terms Ω^n a connection one-form, he shows that indeed Ω^n is invariant under a gauge transformation. This is because Ω is the difference of two connection one-forms, namely the connection one-form A^n and another connection one-form P^n which also depends on the initial state vector. As pointed out by one of the referees, it would be interesting to see whether Pati's results can be generalized to the non-Abelian case. Clearly, if one chooses a basis in the degeneracy subspace $\mathcal{H}_{\lambda_n}(t)$ in which the non-Abelian noncyclic phase factor $\check{\Gamma}^n(t)$ is diagonal, then the diagonal elements may be treated as Abelian noncyclic phase factors and Pati's results may be used to express the corresponding phase angles in the form (96) provided that $\check{\Gamma}^n(t)$ (alternatively $w^n(t)$) is invertable. In such a basis, $\check{\Gamma}^n = \mathrm{diag}(g_1\mathrm{e}^{\mathrm{i}\check{\gamma}^1_n}, \ldots, g_{l_n}\mathrm{e}^{\mathrm{i}\check{\gamma}^{l_n}_n}) =: G^n\mathrm{e}^{\mathrm{i}S_n}$, where 'diag(...)' stands for a diagonal matrix with diagonal elements (...), g_ℓ and $\check{\gamma}^\ell_n$ are real, $G^n := \mathrm{diag}(g_1, \ldots, g_{l_n})$, and $S_n := \mathrm{diag}(\check{\gamma}^1_n, \ldots, \check{\gamma}^{l_n}_n)$. In view of Pati's results, one may find an appropriate diagonal matrix of one-forms $\check{\Omega}^n$ such that $S_n = \int_{\mathcal{C}} \check{\Omega}^n$. Furthermore, $\check{\Omega}^n$ will have the form $A^n - P^n$ for some matrix of one-forms P^n. Now one may postulate that P^n is a connection one-form, so that under a gauge transformation $\check{\Omega}^n$ transforms covariantly. This, together with the form of the diagonal elements of P^n which is given by Pati [8], is sufficient to obtain $\check{\Omega}^n$ in an arbitrary basis. A complete generalization of Pati's results to the non-Abelian case would require a set of properly scaled state vectors $|\chi_n, a; \theta\rangle$ satisfying $\check{\Omega}^n_{ab} = \mathrm{i}\langle\chi_n, a; \theta|d|\chi_n, b; \theta\rangle$. The explicit form of the vectors $|\chi_n, a; \theta(t)\rangle$ is not known to the author.

(3) The function F introduced in section 7 may be used to relate the adiabatic and nonadiabatic Berry connection one-forms. Namely, the nonadiabatic connection one-form $A^n[\theta]$ is the pullback [35, 36] of the adiabatic connection one-form $A^n_\circ[R]$, provided that F is differentiable. This has been originally pointed out in [19]. However, in [19] the existence of F was assumed based on the evidence provided by the study of a magnetic dipole interacting with a precessing magnetic field. In this paper, we have used the theory of dynamical invariants to establish the existence of F for a large class of quantum systems. In particular, it is not difficult to see that they exist for the systems possessing a dynamical group. This allows for the application of the holonomy interpretation of the nonadiabatic geometric phase using a fibre bundle which has the parameter space $\mathcal{M}$ of the invariant as its base space [19]. This bundle is the pullback bundle $F_*(\lambda)$ of the $U(\mathcal{N})$ bundle λ used in the holonomy interpretation of the adiabatic geometric phase [19,36]. Note, however, that the function F depends on the adiabaticity parameter [27]. In general, there are certain values of the adiabaticity parameter for which F becomes ill-defined or discontinuous. The above constructions are valid only for those values of the adiabaticity parameter for which F is a differentiable function.

(4) In our derivation of the noncyclic geometric phase, we assume that the Hamiltonian is a Hermitian operator. The generalization of our results to non-Hermitian Hamiltonians is straightforward. One needs the machinery of the non-Hermitian dynamical invariants and their biorthonormal eigenvectors to obtain a non-Hermitian analogue of the noncyclic geometric phase.

References

[1] Berry M V 1984 *Proc. R. Soc.* **392** 45
[2] Aharonov Y and Anandan J 1987 *Phys. Rev. Lett.* **58** 1593
Anandan J and Aharonov Y 1988 *Phys. Rev.* D **38** 1863
[3] Samuel J and Bhandari R 1988 *Phys. Rev. Lett.* **60** 2339
[4] Zak J 1989 *Europhys. Lett.* **9** 615
[5] Aitchison I J R and Wanelik K 1992 *Proc. R. Soc.* A **439** 25
[6] Mukunda N and Simon R 1993 *Ann. Phys.* **228** 205
[7] Pati A K 1995 *J. Phys. A: Math. Gen.* **28** 2087
[8] Pati A K 1998 *Ann. Phys.* **270** 178
[9] de Polavieja G G and Sjöqvist E 1998 *Am. J. Phys.* **66** 431
[10] Wu Y-S and Li H-Z 1988 *Phys. Rev.* B **38** 11 907
[11] Weinfurter H and Badurek G 1990 *Phys. Rev. Lett.* **64** 1318
[12] Christian J and Shimony A 1993 *J. Phys. A: Math. Gen.* **26** 5551
[13] Wagh A G and Rakhecha V C 1995 *Phys. Lett.* A **197** 107
Wagh A G and Rakhecha V C 1995 *Phys. Lett.* A **197** 112
[14] Wagh A G, Rakhecha V C, Fischer P and Ioffe A 1998 *Phys. Rev. Lett.* **81** 1992
[15] Lewis H R Jr and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458
[16] Morales D A 1988 *J. Phys. A: Math. Gen.* **21** L889
[17] Monteoliva D B, Korsch H J and Núnes J A 1994 *J. Phys. A: Math. Gen.* **27** 6897
[18] Mostafazadeh A 1998 *J. Phys. A: Math. Gen.* **31** 9975
[19] Mostafazadeh A and Bohm A 1993 *J. Phys. A: Math. Gen.* **26** 5473
[20] Anandan J 1988 *Phys. Lett.* A **133** 171
[21] Pressley A and Segal G 1990 *Loop Groups* (Oxford: Oxford University Press)
[22] Mostafazadeh A 1996 *J. Math. Phys.* **37** 1218
[23] Wilczek F and Zee A 1984 *Phys. Rev. Lett.* **52** 2111

[24] Zee A 1988 *Phys. Rev.* A **38** 1
[25] Born M and Fock V 1928 *Z. Phys.* **51** 165
[26] Kato T 1950 *J. Phys. Soc. Japan* **5** 435
[27] Mostafazadeh A 1997 *Phys. Rev.* A **55** 1653
[28] Mostafazadeh A 1997 *J. Phys. A: Math. Gen.* **30** 7525
[29] Mead C A 1987 *Phys. Rev. Lett.* **59** 161
[30] Avron J E, Sadun L, Segert J and Simon B 1988 *Phys. Rev. Lett.* **61** 1329
[31] Avron J E, Sadun L, Segert J and Simon B 1989 *Commun. Math. Phys.* **124** 595
[32] Tycho R 1987 *Phys. Rev. Lett.* **58** 2281
[33] Rabi I I, Ramsey N F and Schwinger J 1954 *Rev. Mod. Phys.* **26** 167
[34] Mostafazadeh A 1997 *J. Math. Phys.* **38** 3489
[35] Nakahara M 1990 *Geometry, Topology and Physics* (New York: Adam Hilger)
[36] Bohm A and Mostafazadeh A 1994 *J. Math. Phys.* **35** 1463